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A NEW SOLUTION FOR MECHANISMS INCLUDING COULOMB FRICTION

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This paper presents a new solution method for the analysis of multibody systems with Coulomb friction. Complete equations of motion and reaction forces are derived by using Lagrangian formulation and the traditional friction circle concepts for the analysis of Coulomb friction. The numerical solutions by the new method, as well as the conventional method for comparision, are illustrated. The new method saves considerably the computer execution time to solve equations of motion and reaction forces compared with the conventional method for same accuracy. The higher the coefficient of friction is, the more the computation time of the conventional method is needed but the computation time of the new method is nearly independent on the coefficient of friction.

Key Words : Coulomb Friction, Kinematic Constraint, Multibody System, Translational Joint, Revolute Joint, Constraint Force, Radius of Friction Circle, Lagrange Multiplier.

1. INTRODUCTION

Machines are composed of many components and several complex mechanisms. These multibody systems consist of rigid bodies interconnected by springs, dashpots, servomotors, bearings and several joints. Hence, bodies of machines do not move independently and have kinematic constraints. Forces and torques induced by friction act on these mechanical joints and they depend not only on the motions of the system but also on the constraint or reaction forces in the system. For the dynamical analysis of machines and mechanisms, the mechanical constraints or joints of multibody systems are often modeled as ideal ones without any friction for simplicity of analysis and its computation. However, the increasing demands on higher precision and speed of machines require an improved analysis of their dynamical behaviours and this means also an increasing accuracy of their mechanical models including the effects of friction in complex multibody systems. As for associated works, there are(Imam, 1981) treating the traditional friction circle concepts and (Haug, 1986 ; Schiehlen, 1983 ; Wu, 1986) using the Lagrange multiplier method and(Rooney, 1982 ; Tomlinson, 1979) discussing Coulomb friction in general. An additional detailed refences are given in (Imam, 1981) and (ed. Haug, 1984).

This paper presents a new solution method for the analysis of multibody systems with Coulomb friction. Complete equations of motion and reaction forces are derived by using Lagrangian formulation and the traditional friction circle concepts for the analysis of Coulomb friction. Computation time and accuracy of the new method are investigated and the numerical solutions by the new method, as well as the conventional method for comparision, are illustrated by its applications to slider-crank mechanism.

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2. EQUATIONS OF MOTION WITH COULOMB FRICTION

In either spatial or planar system dynamics, a set of generalized coordinates of position and orientation is defined as

$$q=[q_1,q_2,\cdots,q_n]^T \tag{1}$$

for n degrees of freedom system.

For planar system dynamics, the vector q is partitioned into subvectors of three generalized coordinates that locate and orient each body in the plane. In the case of spatial motion, q is partitioned into subvectors of six or seven generalized coordinates(using Euler parameters) that locate and orient each body in space.

Physical joints such as spherical joint, revolute joint and translational joint, etc are defined by a set of algebraic constraint equations of the form,

$$\phi(q,t)=[\phi_1(q,t),\phi_2(q,t),\cdots,\phi_s(q,t)]^T=0 \tag{2}$$

if there are s constraints. Where t is time.

Since Eq. (2) must hold throughout an entire interval of time and the generalized coordinate vector q is a function of time, the chain rule of differentiation may be used to differentiate Eq. (2) to obtain the velocity and acceleration equations.

$$\phi_q\dot{q}=-\phi_t \tag{3}$$

$$\phi_q \ddot{q} = -[(\phi_q \dot{q})_q \dot{q} + 2\phi_{tq}\dot{q} + \phi_{tt}] \equiv \gamma \tag{4}$$

Where subscript denotes partial diffentiation and $\phi_q = \partial\phi_i/\partial q_i$ is a $(s \times n)$matrix.

Equations (2) ~ (4) comprise the system of kinematic equations for the mechanical system. The equations of motion of the system may be written using either a Lagrangian or Newton-Euler formulation, with Lagrange multipliers λ to account for constraints, in the form

$$M\ddot{q} + \phi_q{}^T\lambda = Q^e + Q^* \tag{5}$$

Where M is the mass matrix, λ the vector of Lagrange multipliers, Q^e the vector of generalized forces, and Q^* is a vector of quadratic terms in velocities whose coefficients are dependent on the generalized coordinates. Eqs. (2) and (5) comprise a mixed system of differential and algebraic equations that govern dynamics of the system. Combining Eqs. (4) and (5), a set of matrix equations is obtained that determines accelerations and Lagrange multipliers.

$$\begin{bmatrix} M & \phi_q^T \\ \phi_q & 0 \end{bmatrix} \begin{matrix} \ddot{q} \\ \lambda \end{matrix} = \begin{bmatrix} Q^e + Q^* \\ \gamma \end{bmatrix} \tag{6}$$

If the generalized coordinate vector q is divided into two parts of position and orientation components, then

$$\begin{aligned} \ddot{q} &= [\ddot{r}^T \ \dot{\omega}'^T]^T \\ \phi_q &= [\phi_r \ \phi_{\pi'}] \\ M &= \begin{bmatrix} m & 0 \\ 0 & J' \end{bmatrix} \end{aligned} \tag{7}$$

Where $\ddot{r}$, ϕ_r and m are the components corresponding to position generalized coordinates and $\dot{\omega}', \phi_{\pi'}$ and J' are the components corresponding to orientation generalized coordinates.

In a similar manner, $Q^e + Q^*$ can be partitioned into two parts of external force terms F_r and torque terms $n' - \tilde{\omega}' J' \omega'$; *i,e*

$$Q^e + Q^* = \{F_r^T, (n' - \tilde{\omega}' J' \omega')^T\}^T \tag{8}$$

Where F_r is the total force vector, n' is the moment vector of external force with respect to the origin of the body fixed reference frame, J' is the inertia matrix in the body fixed reference frame and $\tilde{\omega}'$ is the skew-symmetric matrix of angular velocity vector ω'.

Inserting Eqs. (7), (8) into Eq. (6), one obtains

$$\begin{bmatrix} m & 0 & \phi_r^T \\ 0 & J' & \phi_{\pi'}{}^T \\ \phi_r & \phi_{\pi'} & 0 \end{bmatrix} \begin{Bmatrix} \ddot{r} \\ \dot{\omega}' \\ \lambda \end{Bmatrix} = \begin{bmatrix} F_r \\ n' - \tilde{\omega}' J' \omega' \\ \gamma \end{bmatrix} \tag{9}$$

The Lagrange multipliers λ in Eq.(9) determine the reaction forces in the kinematic constraints of the system. Joint reaction force can be derived as a function of λ

If the translational motion occurs, the frictional force F_f by the joint reaction force can be also expressed as a function of Lagrange multipliers $\lambda, F_f(\lambda)$ and if the motion is rotation as in the cases of revolute and spherical joints, the frictional torque T_f induced by Coulomb friction can be obtained as $T_f(\lambda)$.

The frictional force F_f and the frictional torque T_f may be inserted in Eq. (9) as additional force and moment to obtain.

$$\begin{bmatrix} m & 0 & \phi_r^T \\ 0 & J' & \phi_{\pi'}{}^T \\ \phi_r & \phi_{\pi'} & 0 \end{bmatrix} \begin{Bmatrix} \ddot{r} \\ \dot{\omega}' \\ \lambda \end{Bmatrix} = \begin{bmatrix} F_r + F_f(\lambda) \\ n' - \tilde{\omega}' J' \omega' + T_f(\lambda) \\ \gamma \end{bmatrix} \tag{10}$$

This is the conventionally known mixed system of differential-algebraic equations of motion of multi-body system with Coulomb friction and is used to solve $\ddot{r}, \dot{\omega}'$, and λ. Note that the Lagrange multipliers λ appear on both left and right-hand sides of Eq. (10). Recalling that λ determine the magnitude of reaction forces in the joints, one may infer that the frictional forces and torques influence reaction forces and torques reversely.

In the following, F_f and T_f of Eq. (10) are derived explicitly for two basic joints, revolute and translational, and they are expressed in convenient forms for application to the new solution method.

2.1 Revolute Joint

Figure 1 shows a member i connected to a member j by a revolute joint with radius R. Members i and j have the angular velocities ω_i and ω_j respectively as shown. The reaction force $-F$ of member j on member i acts at the point of contact P. This force has a component N normal to the two surfaces and a second component F_f tangent to the surfaces developed by the Coulomb sliding where $F_f = \mu N$ and $\mu = \tan(\phi)$ is the coefficient of friction. F_x and F_y are the x and y-directional components of the reaction force F and each of them is an element of Lagrange multipliers λ.

The reaction force $-F$ produces a friction torque $T_f(\lambda)$ about the center of the revolute joint which acts on member i in counter-clockwise direction.

$$\begin{aligned} T_f &= -\text{sign}(\omega_i - \omega_j) FR\sin(\phi) \\ &= -\frac{\mu}{\sqrt{1+\mu^2}} R\,\text{sign}(\omega_i - \omega_j)\sqrt{F_x{}^2 + F_y^2} \\ &= -\frac{\mu}{\sqrt{1+\mu^2}} R\,\text{sign}(\omega_i - \omega_j)(F_x\cos(\theta_{xy}) + F_y\sin(\theta_{xy})) \\ &= -[A_{tf}(\lambda)]\lambda \end{aligned} \tag{11}$$

Where θ_{xy} indicates the direction of the reaction force F and $\text{sign}(\omega_i - \omega_j) = (\omega_i - \omega_j)/|\omega_i - \omega_j|$, $\cdot$ $A_{tf}(\lambda)$ is a $(v \times s)$ matrix, the elements of which are functions of μ, ω_i, ω_j, R and θ_{xy}. Here, v denotes the number of generalized coordinates of orientation. An equal and opposite torque $-T_f(\lambda)$ acts on member j to oppose the rotation. The term $r_f = R\sin(\phi)$ is called the radius of the well-known friction circle.

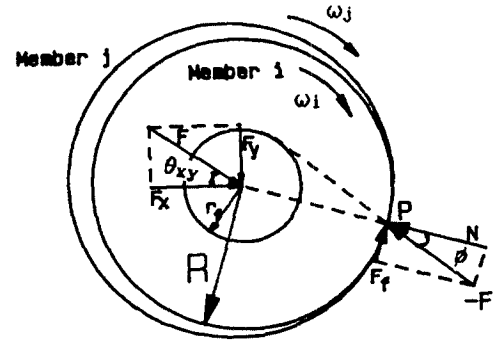

Fig. 1 Revolute joint

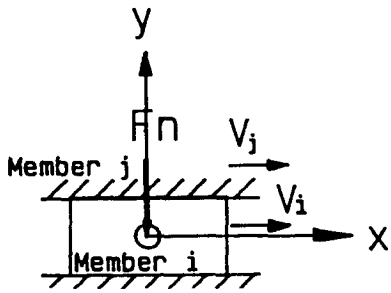

Fig. 2 Translational joint

2.2 Translational Joint

Figure 2 shows a translational joint that connects members i and j. Members i and j have the velocities V_i and V_j respectively. The reaction force F_n normal to the sliding surfaces is also an element of Lagrange multipliers λ

The friction force is obtained as follows;

$$\begin{aligned} F_f(\lambda) &= -\text{sign}(V_i - V_j)\mu|Fn| \\ &= -\text{sign}(V_i - V_j)\mu\frac{Fn}{|Fn|}Fn \\ &= -[A_f(\lambda)]\lambda \end{aligned} \tag{12}$$

Where $A_f(\lambda)$ is a $(u \times s)$ matrix which is determined by μ, V_i, V_j and F_n. Here, u denotes the number of generalized coordinates of position and $u+v=n$.

Inserting Eqs. (11), (12) into Eq. (10) and rearranging, one obtains the following matrix Eq. (13).

$$\begin{bmatrix} m & 0 & \phi_r^T + A_f(\lambda) \\ 0 & J' & \phi_{\pi'}^T + A_{tf}(\lambda) \\ \phi_r & \phi_{\pi'} & 0 \end{bmatrix} \begin{Bmatrix} \dot{r} \\ \dot{\omega}' \\ \lambda \end{Bmatrix} = \begin{Bmatrix} F_r \\ n' - \tilde{\omega}' J' \omega' \\ \gamma \end{Bmatrix} \tag{13}$$

Note that Eq. (13) is different from Eq. (10), the Lagrange multipliers λ appearing only on the left-hand side of Eq. (13) and this fact provides the basis of the numerical algorithm for a new solution for mechanisms including Coulomb friction.

3. NUMERICAL ALGORITHM AND SIMULATION RESULTS

3.1 Numerical Algorithm

(1) Conventional numerical algorithm

Equation (10) is the basic formulation for conventional numerical algorithm. The solutions for $\dot{r}, \dot{\omega}'$ and λ of the preceding step are substituted into the right-hand side of Eq. (10) and new values of $\dot{r}, \dot{\omega}'$ and λ are calculated thereby. This process is repeated until convergence for $\dot{r}, \dot{\omega}'$ and λ is obtained.

This numerical algorithm can be expressed as follows;

$$\begin{bmatrix} m & 0 & \phi_r^T \\ 0 & J' & \phi_{\pi'}^T \\ \phi_r & \phi_{\pi'} & 0 \end{bmatrix} \begin{Bmatrix} \dot{r} \\ \dot{\omega}' \\ \lambda_i \end{Bmatrix} = \begin{bmatrix} F_r + F_f(\lambda_{i-1}) \\ n' - \tilde{\omega}' J' \omega' + T_f(\lambda_{i-1}) \\ \gamma \end{bmatrix} \tag{14}$$

Where i is iteration number of computation.

(2) New numerical algorithm

New numerical algorithm is accomplished by taking advantage of the Eq. (13). The values of $\dot{r}, \dot{\omega}'$ and λ obtained at the preceding step are substituted into the left-hand matrix of Eq. (13) and new values of $\dot{r}, \dot{\omega}'$ and λ are calaulated. This process is repeated until convergence is obtained.

This new numerical algorithm can be represented as the following Eq. (15).

$$\begin{bmatrix} m & 0 & \phi_r^T + A_f(\lambda_{i-1}) \\ 0 & J' & \phi_{\pi'}^T + A_{tf}(\lambda_{i-1}) \\ \phi_r & \phi_{\pi'} & 0 \end{bmatrix} \begin{Bmatrix} \dot{r} \\ \dot{\omega}' \\ \lambda_i \end{Bmatrix} = \begin{bmatrix} F_r \\ n' - \tilde{\omega}' J' \omega' \\ \gamma \end{bmatrix} \tag{15}$$

In practical numeric calculations, the values of $\dot{r}, \dot{\omega}'$ and λ are determined first by solving the Eq. (9) for frictionless case and they are used as the starting values for iteration to solve the Eqs. (14) and (15).

3.2 Simulation Results

To examine the effectiveness and accuracy of this new method, a slider-crank mechanism of Fig. 3 is chosen for its application. This mechanism has three revolute joints of radius of 2.5cm and one translational joint. The poston has cross sectional area of 30cm². Pressure shown in Fig. 4 was applied on the piston, its maximum value being 65.8Kg/cm². Parameters for this mechanism are given in Table 1.

The rotational speed of crankshaft was kept at 2000rpm throughout the whole computations. The equations of both (14) and (15) were solved for each 5 degree increment of the

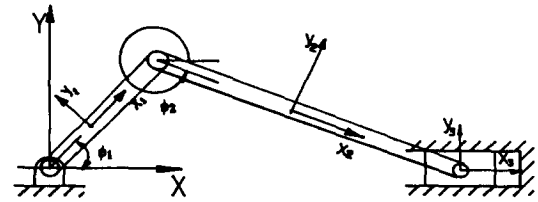

Fig. 3 Slider-crank mechanism

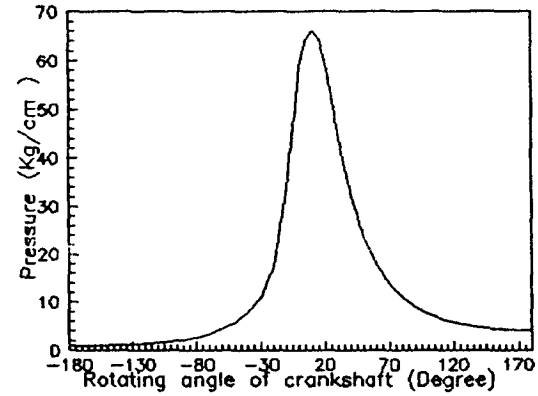

Fig. 4 Piston pressure

Table 1 Slider-crank mechanism parameters

Body No.	Mass (kg)	Moment of inertia (kg cm²)	Length (cm)
1	2.26	54.4	7.62
2	3.36	480.0	28.6
3	2.72	50.0	10.0

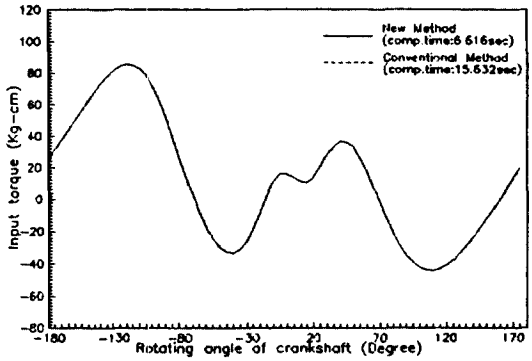

Fig. 5 Input torque ($\mu=0.3$ for translational and revolute joint, 2000rpm, convergence criteria : $|X_i-X_{i-1}|<10^{-5}$)

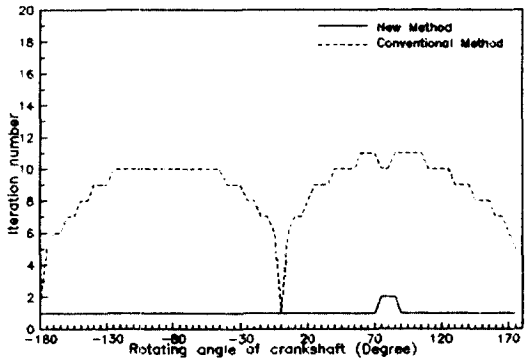

Fig. 6 Iteration number ($\mu=0.3$ for translational joint and $\mu=0.0$ for revolute joint,2000rpm, convergence criteria : $|X_i-X_{i-1}|<10^{-5}$)

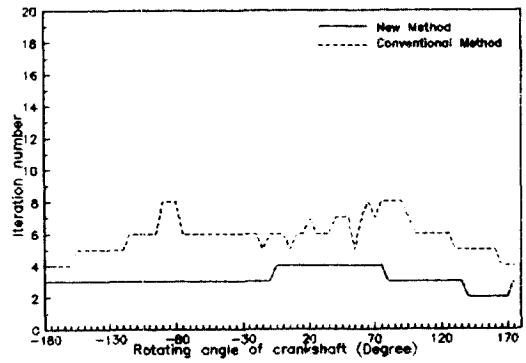

Fig. 7 Interation number ($\mu=0.0$ for translational joint and $\mu=0.3$ for revolute joint, 2000rpm, convergence criteria : $|X_i-X_{i-1}|<10^{-5}$)

crankshaft rotation untill the convergence reached within the prescribed criteria. The iteration number and the time of computation required for each 5 degree increment of the crankshaft rotation were counted. The computation was performed by employing LEQIF (a subroutine of IMSL) and the computer was CYBER 835.

Figure 5 shows input torque calculated by both the conventional and the new method. The coefficient of friction was assumed to be 0.3 and the convergence criteria was set on $|X_i-X_{i-1}|<10^{-5}$. The conventional method required 775 iteration times and 15.632 sec for total computation but the new

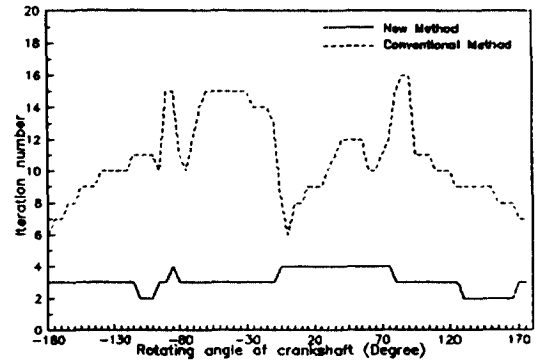

Fig. 8 Iteration number ($\mu=0.3$ for translational and revolute joint, 2000rpm, convergence criteria : $|X_i-X_{i-1}|<10^{-5}$)

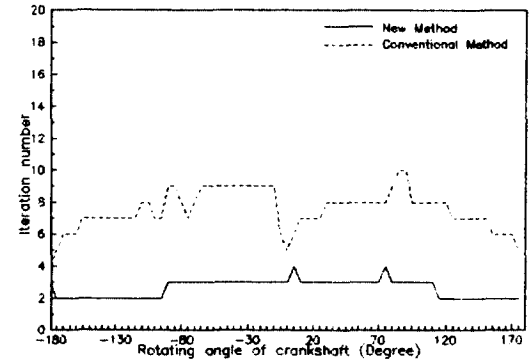

Fig. 9 Iteration number ($\mu=0.1$ for translational and revolute joint, 2000rpm, convergence criteria : $|X_i-X_{i-1}|<10^{-5}$)

Table 2 Total computation iteration number

Coefficient of friction		Conventional method	New method
Translatjonal Joint only	0.05	402	75
	0.10	465	75
	0.15	513	75
	0.20	554	75
	0.25	594	75
	0.30	629	75
Revolute joint only	0.05	320	179
	0.10	346	186
	0.15	381	194
	0.20	390	196
	0.25	406	203
	0.30	421	226
Translational and revolute joint	0.05	468	180
	0.10	542	188
	0.15	612	195
	0.20	664	200
	0.25	722	206
	0.30	775	223

(unit : iteration number of computation)

method 223 iteration times and 6.616 sec only.

Figure 6 through Fig. 9 show the iteration number required to solve the equation of motion for each 5 degree increment of the crankshaft rotation by both the conventional

Table 3 Total computation time

Coefficient of friction		Conventional method	New method
Translational joint only	0.05	9.529	4.107
	0.10	10.612	4.103
	0.15	11.410	4.102
	0.20	12.124	4.114
	0.25	12.803	4.107
	0.30	13.383	4.107
Revolute joint only	0.05	8.139	5.887
	0.10	8.574	5.998
	0.15	9.138	6.175
	0.20	9.273	6.198
	0.25	9.574	6.295
	0.30	9.810	6.685
Translational and revolute joint	0.05	10.607	5.879
	0.10	11.808	6.010
	0.15	12.963	6.124
	0.20	13.824	6.209
	0.25	14.720	6.335
	0.30	15.632	6.616

(unit : CP seconds execution time of CYBER 835)

and the new method for various conditions of friction. In the case of Fig. 6 where the friction exists in translational constraint only, the new method reaches the convergence criteria in one time of iteration through almost whole rotation of the crankshaft, whereas the conventional method reaches in 10 and more times of iteration at some parts of rotation of crankshaft.

Table 2 and Table 3 show the total computation iteration number and the total computation time respectively required by the conventional and the new method for various conditions of friction.

Close examination of these tables reveals that this new method saves considerably the computation time to solve the equations of motion as compared with the conventional method for same accuracy and that the higher the coefficient of friction is, the more the computation time of the conventional method is needed but the computation time of the new method is nearly independent on the coefficient of friction.

4. CONCLUSIONS

Complete equations of motion and reaction forces are established for dynamical analysis of multibody systems by using Lagrangian formulation and applying the traditional friction circle concepts to consider the effects of Coulomb friction. A new solution method of the equations with its numerical algorithm is presented and examined on its effectiveness.

The results obtained are as follows:

(1) The new method saves considerably the computation time to solve the equations of motion and reaction forces compared with the conventional method for same accuracy.

(2) The higher the coefficient of friction is, the more the computation time of the conventional method is needed, but the computation time of the new method is nearly independent on the coefficient of friction.

In the present paper the new solution method was applied to a planar mechanism with simple constraints for demonstration, but its applications can be extended to systems with more complex joints such as spherical joint and its effectiveness may be displayed more remarkably in analysis of large and complex mechanisms.

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